

# A Modified Method for Deriving Self-Conjugate Dirac Hamiltonians in Arbitrary Gravitational Fields and Its Application to Centrally and Axially Symmetric Gravitational Fields

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## Abstract

We have proposed previously a procedure for constructing self-conjugate Hamiltonians  $H_\eta$  in the  $\eta$ -representation with flat scalar product to describe the dynamics of Dirac particles in arbitrary gravitational fields. In this paper, we prove that for block-diagonal metrics the quantities  $H_\eta$  can be obtained in particular using “reduced” parts of Dirac Hamiltonians, i.e. expressions for Dirac Hamiltonians derived at use of tetrad vectors in the Schwinger gauge without or with small numbers of summands with bispinor connectivities. With a glance of these results we propose the modified method of construction of Hamiltonians in the  $\eta$ -representation with a significantly smaller amount of required calculations. Using the proposed procedure, in this paper we for the first time find self-conjugate Hamiltonians for a number of metrics, including the Kerr metric in coordinates of the Boyer-Lindquist, the Eddington-Finkelstein, Finkelstein-Lemaitre, Kruskal metrics, and metrics of the Clifford torus geometry.

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# 1. Introduction

In [1] we proposed a method for constructing self-conjugate Hamiltonians  $H_\eta$  in the  $\eta$ -representation with flat scalar product to describe the dynamics of Dirac particles in arbitrary gravitational fields.

Using the algorithm proposed in [1] we derived Hamiltonians in the  $\eta$ -representation for the Schwarzschild metrics and Friedmann-Robertson-Walker cosmological models. When applying the algorithm to the Kerr metric, however, we faced the necessity of performing a large volume of cumbersome calculations to find Christoffel symbols, bispinor connectivities etc. and also large algebraic transformations of incipient expressions. A situation occurred, in which the algorithm proposed in [1] is fundamentally correct, but in some cases it was non-constructive (such as Kerr metric).

We made attempts to simplify the algorithm [1]. We first proved the theorem, according to which a Hamiltonian in the  $\eta$ -representation for an arbitrary gravitational field, including a time-dependent one, is a Hermitian part of the initial Dirac Hamiltonian  $\tilde{H}$  derived at use of tetrad vectors in the Schwinger gauge <sup>2</sup>.

$$H_\eta = \frac{1}{2} (\tilde{H} + \tilde{H}^+). \quad (1)$$

Then, for block-diagonal metrics using equality (1), we proved the second theorem, according to which instead of Hamiltonians  $\tilde{H}$  and  $\tilde{H}^+$  in equality (1) can be used their “reduced” parts without summands with bispinor connectivities:

$$H_\eta = \frac{1}{2} (\tilde{H}_{red} + \tilde{H}_{red}^+) + \Delta\tilde{H}. \quad (2)$$

We mean that the block-diagonal metrics are the metric tensors, like

$$g_{\alpha\beta} = \begin{array}{|c|c|c|c|} \hline g_{00} & g_{01} & 0 & 0 \\ \hline g_{01} & g_{11} & 0 & 0 \\ \hline 0 & 0 & g_{22} & 0 \\ \hline 0 & 0 & 0 & g_{33} \\ \hline \end{array}. \quad (3)$$

Apparently, the cases belong to the same kind as (3) when  $g_{01} = 0$ , and also when  $g_{02}$  or  $g_{03}$  are used instead of  $g_{01}$ .

In expression (2)  $\tilde{H}_{red}$  is part of the initial Dirac Hamiltonian, which contains only the mass term and terms with momentum operator components (i.e. with coordinate derivatives).

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<sup>2</sup> We use the notations of paper [1].

The summand  $\Delta\tilde{H}$  in Eq. (2)

$$\Delta\tilde{H} = \frac{1}{4} \left( \frac{\partial\tilde{H}_{\underline{n}0}}{\partial x^p} + \frac{g^{0k}}{g^{00}} \frac{\partial\tilde{H}_{\underline{n}k}}{\partial x^p} \right) \tilde{H}_{\underline{m}}^p S_{\underline{mn}} \quad (4)$$

One can see that  $\Delta\tilde{H}$  is a sufficiently simple expression, sometimes it is not equal to zero for block-diagonal metrics with  $g_{0k} \neq 0$ . For example, metric Kerr in coordinates of the Boyer-Lindquist is concerned this case [2], [3].

Of course, application of equality (2) significantly simplifies the procedure of deriving self-conjugate Hamiltonians in the  $\eta$ -representation. Equalities (1) and (2) are proven in Sec. 3, 4 of this paper. A procedure for deriving self-conjugate Hamiltonians in the  $\eta$ -representation with using of expressions (1), (2) we will name the first and the second variants of modified algorithm [1].

In the second part of the paper we use formulas (1), (2) to find self-conjugate Hamiltonians  $H_\eta$  for the Kerr [2], [3], Eddington-Finkelstein [4], [5], Painleve-Gullstrand [6], [7], Finkelstein-Lemaitre [5], Kruskal [8], [9] and Clifford torus [10] metrics. The self-conjugate Hamiltonians are derived for all these metrics for the first time <sup>3</sup>.

## 2. Reducing the Dirac equation to the form of the Schrödinger equation. An algorithm for getting a self-conjugate Hamiltonian in the $\eta$ -representation

Let us recall the course of corresponding reasoning and introduce notations. Tetrad vectors are defined by the relationships

$$H_{\underline{\alpha}}^\mu H_{\underline{\beta}}^\nu g_{\mu\nu} = \eta_{\underline{\alpha}\underline{\beta}}, \quad (5)$$

where

$$\eta_{\underline{\alpha}\underline{\beta}} = \text{diag}[-1, 1, 1, 1]. \quad (6)$$

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<sup>3</sup>The physically equivalent self-conjugate Hamiltonian for Painleve-Gullstrand metric is derived and studied previously in paper [11].

In addition to the system of tetrad vectors  $H_{\underline{\alpha}}^{\mu}$ , one can introduce three other systems of tetrad vectors,  $H_{\underline{\alpha}\mu}$ ,  $H^{\underline{\alpha}\mu}$ ,  $H_{\underline{\mu}}^{\underline{\alpha}}$ , which differ from  $H_{\underline{\alpha}}^{\mu}$  in the location of the global and local (underlined) indices. The global indices are raised up and lowered by means of the metric tensor  $g_{\mu\nu}$  and inverse tensor  $g^{\mu\nu}$ , and the local indices, by means of the tensors  $\eta_{\underline{\alpha}\underline{\beta}}$ ,  $\eta^{\underline{\alpha}\underline{\beta}}$ .

We assume that quantum-mechanical particle motion is described by the Dirac equation, which is written in the units  $\hbar = c = 1$  as

$$\gamma^{\alpha} \left( \frac{\partial \psi}{\partial x^{\alpha}} + \Phi_{\alpha} \psi \right) - m \psi = 0. \quad (7)$$

Here,  $m$  is the particle mass,  $\psi$  is the four-component ‘column’ bispinor, and  $\gamma^{\alpha}$  are the  $4 \times 4$  Dirac matrices satisfying the relationship

$$\gamma^{\alpha} \gamma^{\beta} + \gamma^{\beta} \gamma^{\alpha} = 2g^{\alpha\beta} E. \quad (8)$$

$E$  in (8) is understood to be an identity  $4 \times 4$  matrix.

The round parentheses in (7) contain a covariant derivative of the bispinor  $\nabla_{\alpha} \psi$ ,

$$\nabla_{\alpha} \psi = \frac{\partial \psi}{\partial x^{\alpha}} + \Phi_{\alpha} \psi. \quad (9)$$

Expression (9) for  $\nabla_{\alpha} \psi$  contains bispinor connectivity  $\Phi_{\alpha}$ , for finding which one should fix some system of tetrad vectors  $H_{\underline{\alpha}}^{\mu}$ , defined by the relations (5). Upon that, the quantity  $\Phi_{\alpha}$  can be expressed through ‘Christoffel’ vector derivatives in the following way (the ‘Christoffel’ derivatives are denoted by a semicolon):

$$\Phi_{\alpha} = -\frac{1}{4} H_{\underline{\mu}}^{\underline{\varepsilon}} H_{\nu \underline{\varepsilon}; \alpha} S^{\mu\nu}. \quad (10)$$

The expression for  $S^{\mu\nu}$  in (10) is defined below, see formulas (14). The bispinor connectivity  $\Phi_{\alpha}$  given by (10) provides invariance of the covariant derivative  $\nabla_{\alpha} \psi$  with respect to the transition from one system of tetrad vectors to another.

In what follows, along with Dirac matrices with global indices  $\gamma^{\alpha}$ , we will use Dirac matrices with local indices  $\gamma^{\underline{\alpha}}$ . The relationship between  $\gamma^{\alpha}$  and  $\gamma^{\underline{\alpha}}$  is given by the expression

$$\gamma^{\alpha} = H_{\underline{\beta}}^{\alpha} \gamma^{\underline{\beta}}. \quad (11)$$

It follows from (11), (8), (5) that

$$\gamma^{\underline{\alpha}} \gamma^{\underline{\beta}} + \gamma^{\underline{\beta}} \gamma^{\underline{\alpha}} = 2\eta^{\underline{\alpha}\underline{\beta}} E. \quad (12)$$

In terms of the matrices  $\gamma^\alpha$ , Dirac equation (7) can be written as follows:

$$H_\mu^\alpha \gamma^\mu \left( \frac{\partial \psi}{\partial x^\alpha} + \Phi_\alpha \psi \right) - m\psi = 0. \quad (13)$$

It is convenient (but it is not necessary) to choose the quantities  $\gamma^\alpha$  so that they have the same form for all local frames of reference. Both  $\gamma^\alpha$  and  $\gamma_\alpha$  can be used to construct a full system of  $4 \times 4$  matrices. The full set is, for example, the set

$$E, \quad \gamma_\alpha, \quad S_{\alpha\beta} \equiv \frac{1}{2} (\gamma_\alpha \gamma_\beta - \gamma_\beta \gamma_\alpha), \quad \gamma_5 \equiv \gamma_0 \gamma_1 \gamma_2 \gamma_3, \quad \gamma_5 \gamma_\alpha. \quad (14)$$

Any system of Dirac matrices provides for several discrete automorphisms. We restrict ourselves to the automorphism

$$\gamma_\alpha \rightarrow \gamma_\alpha^+ = -D\gamma_\alpha D^{-1}. \quad (15)$$

The matrix  $D$  will be called anti-Hermitizing.

It follows from (7) that the initial Hamiltonian has the following form:

$$H = -\frac{im}{(-g^{00})} \gamma^0 + \frac{i}{(-g^{00})} \gamma^0 \gamma^k \frac{\partial}{\partial x^k} - i\Phi_0 + \frac{i}{(-g^{00})} \gamma^0 \gamma^k \Phi_k. \quad (16)$$

The operator  $H$  (16) has a meaning of the evolution operator of the Dirac particle wave function in the chosen global frame of reference.

Paper [1] formulates the rules of getting a Hamiltonian in the  $\eta$ -representation for a Dirac particle in an arbitrary gravitational field. A-priori information, which is assumed to be known, includes information about the metric tensor  $g_{\alpha\beta}(x)$ , Christoffel symbols  $\left( \begin{smallmatrix} \lambda \\ \alpha\beta \end{smallmatrix} \right)$ , local metric tensor  $\eta_{\underline{\alpha}\underline{\beta}}$  and local Dirac matrices  $\{\gamma_{\underline{\alpha}}\}$ . These rules are the following:

1) For a gravitational field described by the metric  $g_{\alpha\beta}(x)$ , it is necessary to find a system of tetrads  $\{\tilde{H}_\mu^\alpha(x)\}$  satisfying the Schwinger gauge. Recall that in this gauge the components of the tetrads  $\tilde{H}_0^0$ ,  $\tilde{H}_0^k$  are connected with components of the tensor  $g^{\alpha\beta}(x)$  as follows:

$$\tilde{H}_0^0 = \sqrt{-g^{00}}; \quad \tilde{H}_0^k = -\frac{g^{0k}}{\sqrt{-g^{00}}}. \quad (17)$$

$\tilde{H}_{\underline{k}}^0$  components are identically equal to zero,

$$\tilde{H}_{\underline{k}}^0 = 0. \quad (18)$$

In order to find  $\tilde{H}_{\underline{m}}^n$ , we introduce a tensor  $f^{mn}$  with components

$$f^{mn} = g^{mn} - \frac{g^{0m}g^{0n}}{g^{00}}. \quad (19)$$

The tensor  $f^{mn}$  satisfies the condition

$$f^{mn}g_{nk} = \delta_k^m. \quad (20)$$

As  $\tilde{H}_{\underline{m}}^n$  we can use any triplet of three-dimensional vectors satisfying the relationship

$$\tilde{H}_{\underline{k}}^m \tilde{H}_{\underline{k}}^n = f^{mn}. \quad (21)$$

Here and further quantities dependent on tetrads choice are labeled with tilde on the top, if they were calculated in a system of tetrads in Schwinger gauge.

2) In accordance with (16) we write a general expression for the Hamiltonian  $\tilde{H}$ .

$$\tilde{H} = -\frac{im}{(-g^{00})}\tilde{\gamma}^0 + \frac{i}{(-g^{00})}\tilde{\gamma}^0\tilde{\gamma}^k\frac{\partial}{\partial x^k} - i\tilde{\Phi}_0 + \frac{i}{(-g^{00})}\tilde{\gamma}^0\tilde{\gamma}^k\tilde{\Phi}_k. \quad (22)$$

Here:

$$\tilde{\gamma}^\alpha = \tilde{H}_{\underline{\beta}}^\alpha \gamma_{\underline{\beta}}, \quad (23)$$

$$\tilde{\Phi}_\alpha = -\frac{1}{4}\tilde{H}_{\underline{\mu}}^\varepsilon \tilde{H}_{\underline{\nu}\underline{\varepsilon};\alpha} \tilde{S}^{\mu\nu} = \frac{1}{4}\tilde{H}_{\underline{\mu}}^\varepsilon \tilde{H}_{\underline{\nu}\underline{\varepsilon};\alpha} S^{\mu\nu}. \quad (24)$$

3) The expression for the Hamiltonian  $H_\eta$  equals

$$H_\eta = \tilde{\eta}\tilde{H}\tilde{\eta}^{-1} + i\tilde{\eta}\frac{\partial\tilde{\eta}^{-1}}{\partial t}, \quad (25)$$

where the operator  $\tilde{\eta}$  is given by the relationship

$$\tilde{\eta} = (-g_G)^{1/4} (-g^{00})^{1/4}. \quad (26)$$

In Eq. (26), as distinct from the paper [1], it is used only the gravitational part of the determinant  $g_G$ , because of the presence of external

gravitational field. The other part of the determinant, which appears by using of curvilinear coordinates (cylindrical, spherical and etc.), does not take part in procedure of transition in - representation (see the relation (10) in paper [1]).

$$g_G = \frac{g}{k_c}, \quad (27)$$

where  $k_c$  - curvilinearity coefficient, which in general case can have a complicate functional structure. But in some particular cases the view of  $k_c$  can be defined obvious. Thus at fulfillment of conditions of harmonicity of coordinates [12], [13]  $k_c = 1$  - for Cartesian coordinates,  $k_c = r^2$  - for cylindrical coordinates,  $k_c = r^4 \sin^2 \theta$  - for spherical coordinates.

Expressions (25), (26) define the operator  $H_\eta$ , being the sought Hermitian Hamiltonian in the  $\eta$  representation.

Thus,

$$\begin{aligned} H_\eta = & -\frac{im}{(-g^{00})}\tilde{\gamma}^0 + \frac{i}{(-g^{00})}\tilde{\gamma}^0\tilde{\gamma}^k\frac{\partial}{\partial x^k} - i\tilde{\Phi}_0 + \frac{i}{(-g^{00})}\tilde{\gamma}^0\tilde{\gamma}^k\tilde{\Phi}_k \\ & -\frac{i}{4(-g^{00})}\tilde{\gamma}^0\tilde{\gamma}^k\frac{\partial \ln(g_G g^{00})}{\partial x^k} + \frac{i}{4}\frac{\partial \ln(g_G g^{00})}{\partial t}. \end{aligned} \quad (28)$$

### 3. Proving the equality $H_\eta = \frac{1}{2}(\tilde{H} + \tilde{H}^+)$ for an arbitrary gravitational field, including time-dependent one

We start the proof from transforming the right side of relationship (25).

$$H_\eta = \tilde{\eta}\tilde{H}\tilde{\eta}^{-1} + i\frac{\partial \tilde{\eta}}{\partial t}\tilde{\eta}^{-1}. \quad (29)$$

By putting (22), (26) into (25) we obtain:

$$H_\eta = \tilde{H} - \frac{i}{(-g^{00})}(-g^{00})^{1/2}\gamma^0\tilde{\gamma}^k\frac{1}{4}\frac{\partial \ln(g^{00}g_G)}{\partial x^k} + \frac{i}{4}\frac{\partial \ln(g^{00}g_G)}{\partial t}. \quad (30)$$

The next step in the proof is finding an expression for  $\tilde{H}^+$ . For this purpose, we employ relationships (77) from [1]:

$$\tilde{H}^+ = \tilde{\rho}\tilde{H}\tilde{\rho}^{-1} + \tilde{\Delta}, \quad (31)$$

$$\tilde{\rho} = \left(g^{00}g_G\right)^{1/2}, \quad (32)$$

$$\tilde{\Delta} = \frac{i}{2} \frac{\partial \ln(g^{00}g_G)}{\partial t}. \quad (33)$$

We put (32), (33) into (31):

$$\tilde{H}^+ = \tilde{H} - \frac{i}{(-g^{00})} \left(-g^{00}\right)^{1/2} \gamma^0 \tilde{\gamma}^k \frac{1}{2} \frac{\partial \ln(g^{00}g_G)}{\partial x^k} + \frac{i}{2} \frac{\partial \ln(g^{00}g_G)}{\partial t}. \quad (34)$$

Using (22) and (34), we calculate the quantity  $\frac{1}{2}(\tilde{H} + \tilde{H}^+)$ :

$$\begin{aligned} \frac{1}{2}(\tilde{H} + \tilde{H}^+) &= \tilde{H} - \frac{i}{(-g^{00})} \left(-g^{00}\right)^{1/2} \gamma^0 \tilde{\gamma}^k \frac{1}{4} \frac{\partial \ln(g^{00}g_G)}{\partial x^k} + \\ &+ \frac{i}{4} \frac{\partial \ln(g^{00}g_G)}{\partial t}. \end{aligned} \quad (35)$$

By comparing (35) with (30) we conclude that equality (1) is valid:

$$H_\eta = \frac{1}{2}(\tilde{H} + \tilde{H}^+). \quad (36)$$

#### 4. Proving the equality $H_\eta = \frac{1}{2}(\tilde{H}_{red} + \tilde{H}_{red}^+) + \Delta\tilde{H}$ for arbitrary gravitational fields with block-diagonal metrics

The expression for the “reduced” Hamiltonian  $\tilde{H}_{red}$  is derived from expression (22) by deleting the terms with bispinor connectivities. Thus,

$$\tilde{H}_{red} = -\frac{im}{(-g^{00})} \tilde{\gamma}^0 + \frac{i}{(-g^{00})} \tilde{\gamma}^0 \tilde{\gamma}^k \frac{\partial}{\partial x^k}. \quad (37)$$

This expression can also be written as

$$\tilde{H}_{red} = \tilde{H} + i\tilde{\Phi}_0 - \frac{i}{(-g^{00})} \tilde{\gamma}^0 \tilde{\gamma}^k \tilde{\Phi}_k. \quad (38)$$

Taking the Hermitian conjugation from relationship (38), we get

$$\tilde{H}_{red}^+ = \tilde{H}^+ - i\gamma_0 \tilde{\Phi}_0 \gamma_0 + \frac{i}{(-g^{00})^{1/2}} \gamma_0 \tilde{\Phi}_k \tilde{\gamma}^k. \quad (39)$$



It follows from (38), (39) that

$$\left(\tilde{H}_{red} + \tilde{H}_{red}^+\right) = \left(\tilde{H} + \tilde{H}^+\right) - i\gamma_{\underline{0}}\left[\gamma_{\underline{0}}, \tilde{\Phi}_0\right]_+ + \frac{i}{(-g^{00})^{1/2}}\gamma_{\underline{0}}\left[\tilde{\gamma}^k, \tilde{\Phi}_k\right]_+. \quad (40)$$

Considering (36), we obtain

$$H_\eta = \frac{1}{2}\left(\tilde{H}_{red} + \tilde{H}_{red}^+\right) + \frac{i}{2}\gamma_{\underline{0}}\left[\gamma_{\underline{0}}, \tilde{\Phi}_0\right]_+ + \frac{i}{2(-g^{00})^{1/2}}\gamma_{\underline{0}}\left[\tilde{\gamma}^k, \tilde{\Phi}_k\right]_+. \quad (41)$$

Let us introduce the following notations:

$$Y \equiv \frac{-i}{2}\gamma_{\underline{0}}\left[\gamma_{\underline{0}}, \tilde{\Phi}_0\right]_+, \quad (42)$$

$$Z \equiv -\frac{i}{2(-g^{00})^{1/2}}\gamma_{\underline{0}}\left[\tilde{\gamma}^k, \tilde{\Phi}_k\right]_+. \quad (43)$$

After some transformations, we have the following expressions for  $Y$

$$Y = -\frac{i}{4}\tilde{H}_{\underline{m}}^p\tilde{H}_{\underline{np};0}S_{\underline{mn}}, \quad (44)$$

and for  $Z$

$$Z = Z_1 + Z_2 + Z_3 + Z_4, \quad (45)$$

where

$$Z_1 = -\frac{ig^{0k}}{4(-g^{00})^{1/2}}\tilde{H}_{\underline{m}}^k\tilde{H}_{\underline{m}}^\varepsilon\tilde{H}_{\underline{n\varepsilon};k}S_{\underline{mn}}, \quad (46)$$

$$Z_2 = \frac{i}{4(-g^{00})^{1/2}}\tilde{H}_{\underline{m}}^p\tilde{H}_{\underline{0}}^\varepsilon\tilde{H}_{\underline{n\varepsilon};p}S_{\underline{mn}}, \quad (47)$$

$$Z_3 = -\frac{i}{4(-g^{00})^{1/2}}\tilde{H}_{\underline{m}}^p\tilde{H}_{\underline{n}}^\varepsilon\tilde{H}_{\underline{0\varepsilon};p}S_{\underline{mn}}, \quad (48)$$

$$Z_4 = \frac{i}{4(-g^{00})^{1/2}}\tilde{H}_{\underline{m}}^k\tilde{H}_{\underline{p}}^\varepsilon\tilde{H}_{\underline{q\varepsilon};k}\varepsilon_{\underline{mpq}}\gamma_5. \quad (49)$$

In (49)  $\gamma_5 = \gamma_{\underline{0}}\gamma_{\underline{1}}\gamma_{\underline{2}}\gamma_{\underline{3}}$ ,  $\varepsilon_{\underline{mpq}}$  is completely asymmetric tensor of the third order.

Further we will use the relations (3), (5), (17), (18) and also representation  $\{H_{\underline{k}}^m\}$  in diagonal form. The direct calculations show that

$$\Delta\tilde{H} = Y + Z_1 + Z_2 = \frac{1}{4} \left[ \frac{\partial\tilde{H}_{n0}}{\partial x^p} + \frac{g^{0k}}{g^{00}} \frac{\partial\tilde{H}_{nk}}{\partial x^p} \right] \tilde{H}_{\underline{m}}^p S_{\underline{mn}}, \quad (50)$$

$$Z_3 = Z_4 = 0. \quad (51)$$

Thus, for block-diagonal metrics, like (3), it turns out that we can find the Hamiltonian  $H_\eta$  using simply formula (2).

## 5. Centrally symmetric gravitational field

This section presents Hamiltonians in the  $\eta$ -representation for Dirac particles in centrally symmetric gravitational fields at writing the metric in various coordinates.

### 5.1. Schwarzschild metric

We know that the Schwarzschild solution in the coordinates

$$(x^0, x^1, x^2, x^3) \equiv (t, r, \theta, \varphi) \quad (52)$$

is written as

$$ds^2 = - \left(1 - \frac{r_0}{r}\right) dt^2 + \frac{dr^2}{\left(1 - \frac{r_0}{r}\right)} + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2). \quad (53)$$

In (53),  $r_0$  is the gravitational radius ( $r_0 = 2M$ ).

The general expression for  $H_\eta$ , derived in paper [1] and corrected with a glance of (26), is

$$\begin{aligned} H_\eta = im\sqrt{f}\gamma_{\underline{0}} - i\sqrt{f}\gamma_{\underline{0}} \left\{ \gamma_{\underline{1}}\sqrt{f} \left( \frac{\partial}{\partial r} + \frac{1}{r} \right) + \right. \\ \left. \gamma_{\underline{2}} \frac{1}{r} \left( \frac{\partial}{\partial \theta} + \frac{1}{2} \text{ctg } \theta \right) + \gamma_{\underline{3}} \frac{1}{r \cdot \sin \theta} \frac{\partial}{\partial \varphi} \right\} - \frac{i}{2} \frac{\partial f}{\partial r} \cdot \gamma_{\underline{0}} \gamma_{\underline{1}}. \end{aligned} \quad (54)$$

In formula (54),  $f = 1 - \frac{r_0}{r}$ .

It is easy to verify the expression (54) can be found in a comparatively simply way using formula (2), if we take into account that

$$\tilde{H}_{red} = im\sqrt{f}\gamma_{\underline{0}} - i\sqrt{f}\gamma_{\underline{0}} \left\{ \gamma_{\underline{1}}\sqrt{f}\frac{\partial}{\partial r} + \gamma_{\underline{2}}\frac{1}{r}\frac{\partial}{\partial \theta} + \gamma_{\underline{3}}\frac{1}{r \cdot \sin \theta}\frac{\partial}{\partial \varphi} \right\}, \Delta \tilde{H} = 0. \quad (55)$$

In papers [14], [1], the Hamiltonian  $H_\eta$  was also derived for the Schwarzschild metric in isotropic coordinates

$$ds^2 = -\frac{\left(1 - \frac{r_0}{4R}\right)^2}{\left(1 + \frac{r_0}{4R}\right)^2} dt^2 + \left(1 - \frac{r_0}{4R}\right)^4 (dx^2 + dy^2 + dz^2) \quad (56)$$

$$H_\eta = im\frac{\left(1 - \frac{r_0}{4R}\right)}{\left(1 + \frac{r_0}{4R}\right)}\gamma_{\underline{0}} - \frac{\left(1 - \frac{r_0}{4R}\right)}{\left(1 + \frac{r_0}{4R}\right)^3}\gamma_{\underline{0}}\gamma^{\underline{k}}\frac{\partial}{\partial x^k} - i\frac{\left(1 - \frac{r_0}{8R}\right)}{\left(1 + \frac{r_0}{4R}\right)^4}\frac{r_0}{2}\frac{R_k}{R^3}\gamma_{\underline{0}}\gamma^{\underline{k}}. \quad (57)$$

The expression for  $H_\eta$  can be easily derived from formula (2) using

$$\tilde{H}_{red} = im\frac{\left(1 - \frac{r_0}{4R}\right)}{\left(1 + \frac{r_0}{4R}\right)}\gamma_{\underline{0}} - i\frac{\left(1 - \frac{r_0}{4R}\right)}{\left(1 + \frac{r_0}{4R}\right)^3}\gamma_{\underline{0}}\gamma^{\underline{k}}\frac{\partial}{\partial x^k}, \Delta \tilde{H} = 0. \quad (58)$$

## 5.2. Eddington-Finkelstein metric

The Eddington-Finkelstein solution [4], [5] in the coordinates

$$(x^0, x^1, x^2, x^3) \equiv (t, r, \theta, \varphi) \quad (59)$$

is given by

$$g_{\alpha\beta} = \begin{array}{|c|c|c|c|} \hline -\left(1 - \frac{r_0}{r}\right) & \frac{r_0}{r} & 0 & 0 \\ \hline \frac{r_0}{r} & \left(1 + \frac{r_0}{r}\right) & 0 & 0 \\ \hline 0 & 0 & r^2 & 0 \\ \hline 0 & 0 & 0 & r^2 \sin^2 \theta \\ \hline \end{array}, \quad (60)$$

$$g = -r^4 \cdot \sin^2 \theta, g_G = -1 \quad (61)$$

The inverse tensor has the following form:

$$g^{\alpha\beta} = \begin{array}{|c|c|c|c|} \hline -\left(1 + \frac{r_0}{r}\right) & \frac{r_0}{r} & 0 & 0 \\ \hline \frac{r_0}{r} & \left(1 - \frac{r_0}{r}\right) & 0 & 0 \\ \hline 0 & 0 & \frac{1}{r^2} & 0 \\ \hline 0 & 0 & 0 & \frac{1}{r^2 \sin^2 \theta} \\ \hline \end{array}. \quad (62)$$

Table 1: Tetrad vectors  $\tilde{H}_\mu^\alpha$

Tetrad vectors	Tetrad vector components			
$\tilde{H}_0^\alpha$	$\tilde{H}_0^0 = \sqrt{\left(1 + \frac{r_0}{r}\right)}$	$\tilde{H}_0^1 = -\frac{r_0}{r\sqrt{\left(1 + \frac{r_0}{r}\right)}}$	$\tilde{H}_0^2 = 0$	$\tilde{H}_0^3 = 0$
$\tilde{H}_1^\alpha$	$\tilde{H}_1^0 = 0$	$\tilde{H}_1^1 = \frac{1}{\sqrt{\left(1 + \frac{r_0}{r}\right)}}$	$\tilde{H}_1^2 = 0$	$\tilde{H}_1^3 = 0$
$\tilde{H}_2^\alpha$	$\tilde{H}_2^0 = 0$	$\tilde{H}_2^1 = 0$	$\tilde{H}_2^2 = \frac{1}{r}$	$\tilde{H}_2^3 = 0$
$\tilde{H}_3^\alpha$	$\tilde{H}_3^0 = 0$	$\tilde{H}_3^1 = 0$	$\tilde{H}_3^2 = 0$	$\tilde{H}_3^3 = \frac{1}{r \sin \theta}$

Calculations of a “reduced” Hamiltonian using formula (37) give

$$\begin{aligned} \tilde{H}_{red} = & \frac{im}{\sqrt{\left(1 + \frac{r_0}{r}\right)}} \gamma_{\underline{0}} + \frac{ir_0}{r \left(1 + \frac{r_0}{r}\right)} \frac{\partial}{\partial r} - \\ & - \frac{i}{\sqrt{\left(1 + \frac{r_0}{r}\right)}} \gamma_{\underline{0}} \left( \frac{1}{\sqrt{\left(1 + \frac{r_0}{r}\right)}} \gamma_{\underline{1}} \frac{\partial}{\partial r} + \gamma_{\underline{2}} \frac{1}{r} \frac{\partial}{\partial \theta} + \gamma_{\underline{3}} \frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi} \right). \end{aligned} \quad (63)$$

The Hamiltonian in the  $\eta$ -representation is calculated using formula (2) with glance of  $\Delta \tilde{H} = 0$ . We obtain:

$$\begin{aligned} H_\eta = & i\gamma_{\underline{0}} \frac{m}{\sqrt{\left(1 + \frac{r_0}{r}\right)}} - i\gamma_{\underline{0}} \gamma_{\underline{1}} \frac{1}{\left(1 + \frac{r_0}{r}\right)} \times \\ & \times \left( \left( \frac{\partial}{\partial r} + \frac{1}{r} \right) + \frac{r_0}{2r^2} \frac{1}{\left(1 + \frac{r_0}{r}\right)} \right) - \\ & - i\gamma_{\underline{0}} \gamma_{\underline{2}} \frac{1}{\sqrt{\left(1 + \frac{r_0}{r}\right)}} \frac{1}{r} \left( \frac{\partial}{\partial \theta} + \frac{1}{2} \text{ctg} \theta \right) - i\gamma_{\underline{0}} \gamma_{\underline{3}} \frac{1}{\sqrt{\left(1 + \frac{r_0}{r}\right)}} \frac{\partial}{\partial \varphi} + \\ & + i\frac{r_0}{r} \frac{1}{\left(1 + \frac{r_0}{r}\right)} \left( \left( \frac{\partial}{\partial r} + \frac{1}{r} \right) - \frac{1}{2r \left(1 + \frac{r_0}{r}\right)} \right). \end{aligned} \quad (64)$$

### 5.3. Painleve-Gullstrand metric

In this section we find self-conjugate Hamiltonian  $H_\eta$  for Dirac particle in spherically symmetric gravitational field described by Painleve-Gullstrand metric. The Hamiltonian  $H_\eta$  for this metric is calculated first using the algorithm [1] and then using the formulas (1) and (2).

In the  $(t, r, \theta, \varphi)$  coordinates, the Painleve-Gullstrand metric [6], [7] has the following form:

$$\begin{aligned}
g_{\alpha\beta} = & \begin{array}{|c|c|c|c|} \hline -\left(1 - \frac{r_0}{r}\right) & \sqrt{\frac{r_0}{r}} & 0 & 0 \\ \hline \sqrt{\frac{r_0}{r}} & 1 & 0 & 0 \\ \hline 0 & 0 & r^2 & 0 \\ \hline 0 & 0 & 0 & r^2 \sin^2 \theta \\ \hline \end{array} \\
g^{\alpha\beta} = & \begin{array}{|c|c|c|c|} \hline -1 & \sqrt{\frac{r_0}{r}} & 0 & 0 \\ \hline \sqrt{\frac{r_0}{r}} & \left(1 - \frac{r_0}{r}\right) & 0 & 0 \\ \hline 0 & 0 & \frac{1}{r^2} & 0 \\ \hline 0 & 0 & 0 & \frac{1}{r^2 \sin^2 \theta} \\ \hline \end{array}.
\end{aligned} \tag{65}$$

The determinant is

$$g = -r^4 \sin^2 \theta, g_G = -1. \tag{66}$$

Tetrad vectors in the Schwinger gauge:

$$\left. \begin{aligned} \tilde{H}_{\underline{0}}^0 &= 1, & \tilde{H}_{\underline{0}}^1 &= -\sqrt{\frac{r_0}{r}}, & \tilde{H}_{\underline{0}}^2 &= 0, & \tilde{H}_{\underline{0}}^3 &= 0, \\ \tilde{H}_{\underline{1}}^0 &= 0, & \tilde{H}_{\underline{1}}^1 &= 1, & \tilde{H}_{\underline{1}}^2 &= 0, & \tilde{H}_{\underline{1}}^3 &= 0, \\ \tilde{H}_{\underline{2}}^0 &= 0, & \tilde{H}_{\underline{2}}^1 &= 0, & \tilde{H}_{\underline{2}}^2 &= \frac{1}{r}, & \tilde{H}_{\underline{2}}^3 &= 0, \\ \tilde{H}_{\underline{3}}^0 &= 0, & \tilde{H}_{\underline{3}}^1 &= 0, & \tilde{H}_{\underline{3}}^2 &= 0, & \tilde{H}_{\underline{3}}^3 &= \frac{1}{r \cdot \sin \theta} \end{aligned} \right\}. \tag{67}$$

$$\left. \begin{aligned} \tilde{H}_{\underline{00}} &= -1, & \tilde{H}_{\underline{01}} &= 0, & \tilde{H}_{\underline{02}} &= 0, & \tilde{H}_{\underline{03}} &= 0, \\ \tilde{H}_{\underline{10}} &= \sqrt{\frac{r_0}{r}}, & \tilde{H}_{\underline{11}} &= 1, & \tilde{H}_{\underline{12}} &= 0, & \tilde{H}_{\underline{13}} &= 0, \\ \tilde{H}_{\underline{20}} &= 0, & \tilde{H}_{\underline{21}} &= 0, & \tilde{H}_{\underline{22}} &= r, & \tilde{H}_{\underline{23}} &= 0, \\ \tilde{H}_{\underline{30}} &= 0, & \tilde{H}_{\underline{31}} &= 0, & \tilde{H}_{\underline{32}} &= 0, & \tilde{H}_{\underline{33}} &= r \cdot \sin \theta \end{aligned} \right\}. \tag{68}$$

Christoffel symbols :

$$\left. \begin{aligned}
\begin{pmatrix} 0 \\ 00 \end{pmatrix} &= \frac{1}{2r^2} \sqrt{\frac{r_0}{r}} \\
\begin{pmatrix} 0 \\ 01 \end{pmatrix} &= \frac{1}{2} \frac{r_0}{r^2} \\
\begin{pmatrix} 0 \\ 11 \end{pmatrix} &= \frac{1}{2r} \sqrt{\frac{r_0}{r}} \\
\begin{pmatrix} 0 \\ 02 \end{pmatrix} &= \begin{pmatrix} 0 \\ 03 \end{pmatrix} = \begin{pmatrix} 0 \\ 12 \end{pmatrix} = \begin{pmatrix} 0 \\ 13 \end{pmatrix} = \begin{pmatrix} 0 \\ 23 \end{pmatrix} = 0 \\
\begin{pmatrix} 0 \\ 22 \end{pmatrix} &= -\sqrt{r_0 r} \\
\begin{pmatrix} 0 \\ 33 \end{pmatrix} &= -\sqrt{r_0 r} \sin^2 \theta
\end{aligned} \right\}. \quad (69)$$

$$\left. \begin{aligned}
\begin{pmatrix} 1 \\ 00 \end{pmatrix} &= \frac{1}{2} \frac{r_0}{r^2} \left( 1 - \frac{r_0}{r} \right) \\
\begin{pmatrix} 1 \\ 01 \end{pmatrix} &= -\frac{1}{2} \frac{r_0^{3/2}}{r^{5/2}} \\
\begin{pmatrix} 1 \\ 11 \end{pmatrix} &= -\frac{1}{2} \frac{r_0}{r^2} \\
\begin{pmatrix} 1 \\ 02 \end{pmatrix} &= \begin{pmatrix} 1 \\ 03 \end{pmatrix} = \begin{pmatrix} 1 \\ 12 \end{pmatrix} = \begin{pmatrix} 1 \\ 13 \end{pmatrix} = \begin{pmatrix} 1 \\ 23 \end{pmatrix} = 0 \\
\begin{pmatrix} 1 \\ 22 \end{pmatrix} &= -r \left( 1 - \frac{r_0}{r} \right) \\
\begin{pmatrix} 1 \\ 33 \end{pmatrix} &= -r \sin^2 \theta \left( 1 - \frac{r_0}{r} \right)
\end{aligned} \right\}. \quad (70)$$

$$\left. \begin{aligned}
\begin{pmatrix} 2 \\ 12 \end{pmatrix} &= \frac{1}{r} \\
\begin{pmatrix} 2 \\ 00 \end{pmatrix} &= \begin{pmatrix} 2 \\ 01 \end{pmatrix} = \begin{pmatrix} 2 \\ 11 \end{pmatrix} = \begin{pmatrix} 2 \\ 02 \end{pmatrix} = \begin{pmatrix} 2 \\ 03 \end{pmatrix} = \\
&= \begin{pmatrix} 2 \\ 13 \end{pmatrix} = \begin{pmatrix} 2 \\ 22 \end{pmatrix} = \begin{pmatrix} 2 \\ 23 \end{pmatrix} = 0 \\
\begin{pmatrix} 2 \\ 33 \end{pmatrix} &= -\sin \theta \cos \theta
\end{aligned} \right\}. \quad (71)$$



$$\left. \begin{aligned} \begin{pmatrix} 3 \\ 13 \end{pmatrix} &= \frac{1}{r} \\ \begin{pmatrix} 3 \\ 00 \end{pmatrix} &= \begin{pmatrix} 3 \\ 01 \end{pmatrix} = \begin{pmatrix} 3 \\ 11 \end{pmatrix} = \begin{pmatrix} 3 \\ 02 \end{pmatrix} = \\ &= \begin{pmatrix} 3 \\ 12 \end{pmatrix} = \begin{pmatrix} 3 \\ 03 \end{pmatrix} = \begin{pmatrix} 3 \\ 22 \end{pmatrix} = \begin{pmatrix} 3 \\ 33 \end{pmatrix} = 0 \\ \begin{pmatrix} 3 \\ 23 \end{pmatrix} &= \frac{\cos \theta}{\sin \theta} \end{aligned} \right\}. \quad (72)$$

Bispinor connectivities are calculated using the formula (24) and the relations (65) - (72). We obtain:

$$\left. \begin{aligned} \tilde{\Phi}_0 &= \frac{1}{4} \frac{r_0}{r^2} S_{\underline{01}} \\ \tilde{\Phi}_1 &= \frac{1}{4} \frac{1}{r} \sqrt{\frac{r_0}{r}} S_{\underline{01}} \\ \tilde{\Phi}_2 &= -\frac{1}{2} \sqrt{\frac{r_0}{r}} S_{\underline{02}} - \frac{1}{2} S_{\underline{12}} \\ \tilde{\Phi}_3 &= -\frac{1}{2} \sqrt{\frac{r_0}{r}} \sin \theta \cdot S_{\underline{03}} + \frac{1}{2} \sin \theta \cdot S_{\underline{31}} - \frac{1}{2} \cos \theta \cdot S_{\underline{23}} \end{aligned} \right\}. \quad (73)$$

In order to find a Hamiltonian in the  $\eta$ -representation, one should insert  $(-g^{00}) = 1$ , expressions  $\tilde{\gamma}^\alpha = \tilde{H}_{\underline{\mu}}^\alpha \gamma^\mu$  and expressions (73) for  $\tilde{\Phi}_\alpha$  into the primary Dirac Hamiltonian  $\tilde{H}$

$$\tilde{H} = -\frac{im}{(-g^{00})} \tilde{\gamma}^0 + \frac{i}{(-g^{00})} \tilde{\gamma}^0 \tilde{\gamma}^k \frac{\partial}{\partial x^k} - i\tilde{\Phi}_0 + \frac{i}{(-g^{00})} \tilde{\gamma}^0 \tilde{\gamma}^k \tilde{\Phi}_k. \quad (74)$$

As result, we have

$$\begin{aligned} \tilde{H} &= im\gamma_{\underline{0}} - i\gamma_{\underline{0}} \left\{ \gamma_{\underline{1}} \left( \frac{\partial}{\partial r} + \frac{1}{r} \right) + \gamma_{\underline{2}} \frac{1}{r} \left( \frac{\partial}{\partial \theta} + \frac{1}{2} \text{ctg} \theta \right) + \right. \\ &\quad \left. + \gamma_{\underline{3}} \frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi} \right\} + i\sqrt{\frac{r_0}{r}} \frac{\partial}{\partial r} + i\frac{3}{4} \frac{1}{r} \sqrt{\frac{r_0}{r}}. \end{aligned} \quad (75)$$

The operator  $\tilde{\eta}$  for Painleve-Gullstrand metric is

$$\tilde{\eta} = (g_G g^{00})^{\frac{1}{2}} = 1 \quad (76)$$

and therefore  $\eta$  - representation coincides with representation of the Hamiltonian  $\tilde{H}$ .

The Hamiltonian  $\tilde{H}$  (75) is self-conjugate. The validity of Eq. (1) in this case is obvious.

It is also easy to obtain expression (75) using Eq. (2) and taking into account that  $\Delta\tilde{H}$  and  $\tilde{H}_{red}$  can be written as

$$\tilde{H}_{red} = im\gamma_{\underline{0}} - i\gamma_{\underline{0}} \left\{ \gamma_{\underline{1}} \frac{\partial}{\partial r} + \gamma_{\underline{2}} \frac{1}{r} \frac{\partial}{\partial \theta} + \gamma_{\underline{3}} \frac{1}{r \cdot \sin \theta} \frac{\partial}{\partial \varphi} \right\} + i \sqrt{\frac{r}{r_0}} \frac{\partial}{\partial r}. \quad (77)$$

Thereby, for Painleve-Gullstrand metric, one the same Hamiltonian  $H_\eta$  is derived both the standard algorithm and the most simply method with using Eq. (2).

In paper [11], the self-conjugate Hamiltonian for Painleve-Gullstrand is derived with use of tetrad vectors in the Schwinger gauge with set of the local Dirac metrics written in spherical coordinate system.

$$\left. \begin{aligned} \gamma_{\underline{0}} &= \gamma_{\underline{0}} \\ \gamma_{\underline{r}} &= \sin \theta \left[ \gamma_{\underline{1}} \cos \varphi + \gamma_{\underline{2}} \sin \varphi \right] + \gamma_{\underline{3}} \cos \varphi = R\gamma_{\underline{1}}R^{-1} \\ \gamma_{\underline{\theta}} &= \cos \theta \left[ \gamma_{\underline{1}} \cos \varphi + \gamma_{\underline{2}} \sin \varphi \right] - \gamma_{\underline{3}} \sin \varphi = R\gamma_{\underline{2}}R^{-1} \\ \gamma_{\underline{\varphi}} &= \gamma_{\underline{1}} \sin \varphi + \gamma_{\underline{2}} \cos \varphi = R\gamma_{\underline{3}}R^{-1} \end{aligned} \right\}. \quad (78)$$

The set  $\{\gamma_{\underline{r}}, \gamma_{\underline{\theta}}, \gamma_{\underline{\varphi}}\}$  connects with the set  $\{\gamma_{\underline{1}}, \gamma_{\underline{2}}, \gamma_{\underline{3}}\}$  by unitary matrix  $R$

$$\begin{aligned}
R &= R_1 T_1 R_2 T_2 \\
R_1 &= \exp\left(-\frac{\varphi}{2}\right) \gamma_1 \gamma_2; T_1 = \frac{1}{\sqrt{2}} \gamma_5 \gamma_1 (E + \gamma_1 \gamma_2) \\
R_2 &= \exp\left(-\frac{\theta}{2}\right) \gamma_1 \gamma_3; T_2 = \frac{1}{\sqrt{2}} \gamma_5 \gamma_2 (E + \gamma_3 \gamma_1)
\end{aligned} \tag{79}$$

The Hamiltonian of paper [11] one can write as

$$\tilde{H}_D = im\gamma_0 - i\gamma_0 \left\{ \gamma_r \frac{\partial}{\partial r} + \gamma_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \gamma_\varphi \frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi} \right\} + i\sqrt{\frac{r_0}{r}} \left( \frac{\partial}{\partial r} + \frac{3}{4r} \right). \tag{80}$$

The Hamiltonian (75) and (80) are physically equivalent because they are connected by unitary transformation

$$\tilde{H}_D = R \tilde{H} R^{-1}, R^+ = R^{-1}. \tag{81}$$

Generally speaking, the all Hamiltonians in the Schwinger gauge, connected among themselves by matrices of spatial rotation, are physically equivalent. Just this fact is meant by authors [1], when they spoke about uniqueness of the Hamiltonians in  $\eta$  - representation (see, M.Arminjon remarks in [15]).

#### 5.4. Finkelstein-Lemaitre metric

It is of independent interest to study the motion of a Dirac particle in the nonstationary Finkelstein-Lemaitre metric [5], because the time coordinate in this metric coincides with the proper time.

$$ds^2 = -dt^2 + \frac{dr^2}{\left[\frac{3}{2r_0}(r-t)\right]^{2/3}} + \left[\frac{3}{2}(r-t)\right]^{4/3} r_0^{2/3} (d\theta^2 + \sin^2 \theta \cdot d\varphi^2). \tag{82}$$

The determinants are

$$\left. \begin{aligned} g &= -\frac{3}{2} (r-t)^2 r_0^2 \sin^2 \theta \\ g_G &= -\frac{3}{2} (r-t)^2 \frac{r_0^2}{r^4} \end{aligned} \right\}. \quad (83)$$

Non-zero components of tetrad vectors in the Schwinger gauge:

$$\begin{aligned} \tilde{H}_0^0 &= 1, \quad \tilde{H}_1^1 = \left[ \frac{3}{2r_0} (r-t) \right]^{1/3}, \\ \tilde{H}_2^2 &= \frac{1}{\left[ \frac{3}{2} (r-t) \right]^{2/3} r_0^{1/3}}, \quad \tilde{H}_3^3 = \frac{1}{\left[ \frac{3}{2} (r-t) \right]^{2/3} r_0^{1/3} \sin \theta}. \end{aligned} \quad (84)$$

For this metric in Eq. (2)  $\tilde{H} = 0$ .

“Reduced” Hamiltonian:

$$\tilde{H}_{red} = im\gamma_0 - i\gamma_0 \left\{ \tilde{H}_1^1 \gamma_1 \frac{\partial}{\partial r} + \tilde{H}_2^2 \gamma_2 \frac{\partial}{\partial \theta} + \tilde{H}_3^3 \gamma_3 \frac{\partial}{\partial \varphi} \right\}. \quad (85)$$

We substitute (85) into (2) and get

$$\begin{aligned} H_\eta &= im\gamma_0 - i\gamma_0 \left\{ \tilde{H}_1^1 \gamma_1 \left( \frac{\partial}{\partial r} + \frac{1}{r} \right) + \tilde{H}_2^2 \gamma_2 \left( \frac{\partial}{\partial \theta} + \frac{1}{2} \text{ctg } \theta \right) + \right. \\ &\quad \left. + \tilde{H}_3^3 \gamma_3 \frac{\partial}{\partial \varphi} \right\} - \frac{i}{2} \gamma_0 \gamma_1 \frac{\partial \tilde{H}_1^1}{\partial r}. \end{aligned} \quad (86)$$

Hamiltonian (86) is self-conjugate with fairly complicated time dependence.

### 5.5. A Hamiltonian in the $\eta$ -representation for Dirac particles in the Kruskal gravitational field

The Kruskal metric [8] is a further development of the Lemaitre-Finkelstein metric to build the most complete frame of reference for a point-mass field. The solution form given below, in which the frame of reference is synchronous, has been developed by I.D. Novikov [9].

In  $(\tau, R, \theta, \varphi)$  coordinates

$$ds^2 = -d\tau^2 + \left(1 + \frac{R^2}{r_0^2}\right) (1 - \cos \chi)^2 dR^2 + \frac{1}{4} r_0^2 \left(\frac{R^2}{r_0^2} + 1\right) (1 - \cos \chi)^2 (d\theta^2 + \sin^2 \theta d\varphi^2). \quad (87)$$

$$\frac{\tau}{r_0} = \frac{1}{2} \left(\frac{R^2}{r_0^2} + 1\right)^{3/2} (\pi - \chi + \sin \chi). \quad (88)$$

The determinants are

$$g = - \left(1 + \frac{R^2}{r_0^2}\right)^3 (1 - \cos \chi)^6 \frac{1}{16} r_0^4 \sin^2 \theta$$

$$g_G = - \left(1 + \frac{R^2}{r_0^2}\right)^3 (1 - \cos \chi)^6 \frac{1}{16} \frac{r_0^4}{R^4} \quad . \quad (89)$$

Expressions (87), (88) show that metric (87) depends on the radial coordinate  $R$  and on the proper time  $\tau$  by means of parameter  $\eta$ .

Non-zero components of tetrad vectors in the Schwinger gauge:

$$\tilde{H}_0^0 = 1, \quad \tilde{H}_1^1 = \frac{1}{\sqrt{\left(1 + \frac{R^2}{r_0^2}\right) (1 - \cos \chi)}};$$

$$\tilde{H}_2^2 = \frac{2}{r_0 \left(\frac{R^2}{r_0^2} + 1\right) (1 - \cos \chi)}, \quad (90)$$

$$\tilde{H}_3^3 = \frac{2}{r_0 \left(\frac{R^2}{r_0^2} + 1\right) (1 - \cos \chi) \cdot \sin \theta}.$$

“Reduced” Hamiltonian:

$$\tilde{H}_{red} = im\gamma_0 - i\gamma_0 \left\{ \tilde{H}_1^1 \gamma_1 \frac{\partial}{\partial R} + \tilde{H}_2^2 \gamma_2 \frac{\partial}{\partial \theta} + \tilde{H}_3^3 \gamma_3 \frac{\partial}{\partial \varphi} \right\}. \quad (91)$$

In accordance with formula (2) and with glance  $\Delta\tilde{H} = 0$  we have

$$H_\eta = im\gamma_0 - i\gamma_0 \left\{ \tilde{H}_1^1 \gamma_1 \left( \frac{\partial}{\partial R} + \frac{1}{R} \right) + \tilde{H}_2^2 \gamma_2 \left( \frac{\partial}{\partial \theta} + \frac{1}{2} \text{ctg } \theta \right) + \right. \\ \left. + \tilde{H}_3^3 \gamma_3 \frac{\partial}{\partial \varphi} \right\} - \frac{i}{2} \gamma_0 \gamma_1 \frac{\partial \tilde{H}_1^1}{\partial R}. \quad (92)$$

The derivative  $\frac{\partial \tilde{H}_1^1}{\partial R}$  in the last summand of expression (92) should be defined accounting for the dependence  $\chi(R, \tau)$  (see formula (88)).

## 6. Axially symmetric gravitational field

### 6.1. Kerr metric in the Boyer-Lindquist coordinates

The Kerr solution in the Boyer-Lindquist coordinates [3]

$$(x^0, x^1, x^2, x^3) \equiv (t, r, \theta, \varphi) \quad (93)$$

is given by

$$g_{\alpha\beta} = \begin{array}{|c|c|c|c|} \hline -\left(1 - \frac{r_0 r}{\rho^2}\right) & 0 & 0 & -\frac{ar_0 r}{\rho^2} \sin^2 \theta \\ \hline 0 & \frac{\rho^2}{\Delta} & 0 & 0 \\ \hline 0 & 0 & \rho^2 & 0 \\ \hline -\frac{ar_0 r}{\rho^2} \sin^2 \theta & 0 & 0 & \left(r^2 + a^2 + \frac{a^2 r_0 r}{\rho^2} \cdot \sin^2 \theta\right) \cdot \sin^2 \theta \\ \hline \end{array}, \quad (94)$$

$$g = -\rho^4 \cdot \sin^2 \theta, g_G = -\frac{\rho}{r^4}. \quad (95)$$

The inverse tensor has the following form:

$$g^{\alpha\beta} = \begin{array}{c} \\ \\ = \end{array} \begin{array}{|c|c|c|c|} \hline -\frac{1}{\Delta} \cdot \left( r^2 + a^2 + \frac{a^2 r_0 r}{\rho^2} \cdot \sin^2 \theta \right) & 0 & 0 & -\frac{a r_0 r}{\Delta \cdot \rho^2} \\ \hline 0 & \frac{\Delta}{\rho^2} & 0 & 0 \\ \hline 0 & 0 & \frac{1}{\rho^2} & 0 \\ \hline -\frac{a r_0 r}{\Delta \cdot \rho^2} & 0 & 0 & \frac{1}{\Delta \cdot \sin^2 \theta} \left( 1 - \frac{r_0 r}{\rho^2} \right) \\ \hline \end{array} . \quad (96)$$

Here,

$$\left. \begin{array}{l} \Delta \equiv r^2 - r_0 r + a^2 \\ \rho^2 \equiv r^2 + a^2 \cdot \cos^2 \theta \end{array} \right\} . \quad (97)$$

## 6.2. Tetrad vectors in the Schwinger gauge

We will need expressions for tetrad vectors in the Schwinger gauge. The results of calculating the components of tetrad vectors  $\tilde{H}_{\underline{\mu}}^{\alpha}$  are presented in Table 2. Table 3 shows the components of vectors  $\tilde{H}_{\underline{\mu}\alpha}$ .

Table 2: Tetrad vectors  $\tilde{H}_{\underline{\mu}}^{\alpha}$

Tetrad vectors	Tetrad vector components			
$\tilde{H}_{\underline{0}}^{\alpha}$	$\frac{\tilde{H}_{\underline{0}}^0}{\sqrt{(-g^{00})}} =$	$\tilde{H}_{\underline{0}}^1 = 0$	$\tilde{H}_{\underline{0}}^2 = 0$	$\frac{\tilde{H}_{\underline{0}}^3}{\frac{ar_0r}{\rho^2\Delta\sqrt{(-g^{00})}}} =$
$\tilde{H}_{\underline{1}}^{\alpha}$	$\tilde{H}_{\underline{1}}^0 = 0$	$\frac{\tilde{H}_{\underline{1}}^1}{\frac{\sqrt{\Delta}}{\rho}} =$	$\tilde{H}_{\underline{1}}^2 = 0$	$\tilde{H}_{\underline{1}}^3 = 0$
$\tilde{H}_{\underline{2}}^{\alpha}$	$\tilde{H}_{\underline{2}}^0 = 0$	$\tilde{H}_{\underline{2}}^1 = 0$	$\tilde{H}_{\underline{2}}^2 = \frac{1}{\rho}$	$\tilde{H}_{\underline{2}}^3 = 0$
$\tilde{H}_{\underline{3}}^{\alpha}$	$\tilde{H}_{\underline{3}}^0 = 0$	$\tilde{H}_{\underline{3}}^1 = 0$	$\tilde{H}_{\underline{3}}^2 = 0$	$\frac{\tilde{H}_{\underline{3}}^3}{\frac{1}{\sin\theta \cdot \sqrt{\Delta}\sqrt{(-g^{00})}}} =$



Table 3: Tetrad vectors  $\tilde{H}_{\underline{\mu}\alpha}$

Tetrad vectors	Tetrad vector components			
$\tilde{H}_{\underline{0}\alpha}$	$\tilde{H}_{\underline{0}0} = -\frac{1}{\sqrt{(-g^{00})}}$	$\tilde{H}_{\underline{0}1} = 0$	$\tilde{H}_{\underline{0}2} = 0$	$\tilde{H}_{\underline{0}3} = 0$
$\tilde{H}_{\underline{1}\alpha}$	$\tilde{H}_{\underline{1}0} = 0$	$\tilde{H}_{\underline{1}1} = \frac{\tilde{\rho}}{\sqrt{\Delta}}$	$\tilde{H}_{\underline{1}2} = 0$	$\tilde{H}_{\underline{1}3} = 0$
$\tilde{H}_{\underline{2}\alpha}$	$\tilde{H}_{\underline{2}0} = 0$	$\tilde{H}_{\underline{2}1} = 0$	$\tilde{H}_{\underline{2}2} = \rho$	$\tilde{H}_{\underline{2}3} = 0$
$\tilde{H}_{\underline{3}\alpha}$	$\tilde{H}_{\underline{3}0} = -\frac{ar_0 r \cdot \sin \theta}{\rho^2 \sqrt{\Delta} \sqrt{(-g^{00})}}$	$\tilde{H}_{\underline{3}1} = 0$	$\tilde{H}_{\underline{3}2} = 0$	$\tilde{H}_{\underline{3}3} = \sin \theta \cdot \sqrt{\Delta} \sqrt{(-g^{00})}$

### 6.3. Hamiltonian $H_\eta$

First, from formula (37) we get

$$\begin{aligned}
\tilde{H}_{red} &\equiv -\frac{im}{(-g^{00})}\tilde{\gamma}^0 + \frac{i}{(-g^{00})}\tilde{\gamma}^0\tilde{\gamma}^k\frac{\partial}{\partial x^k} = -\frac{im}{(-g^{00})}\tilde{H}_0^0\gamma^0 + \\
&+ \frac{i}{(-g^{00})}\tilde{H}_0^0\gamma^0\left\{\tilde{H}_1^1\gamma^1\frac{\partial}{\partial x^1} + \tilde{H}_2^2\gamma^2\frac{\partial}{\partial x^2} + \tilde{H}_3^3\gamma^3\frac{\partial}{\partial x^3} + \tilde{H}_0^3\gamma^0\frac{\partial}{\partial x^3}\right\} = \\
&= \frac{im}{(-g^{00})}\tilde{H}_0^0\gamma_0 - \frac{i}{(-g^{00})}\tilde{H}_0^0\tilde{H}_1^1\gamma_0\gamma_1\frac{\partial}{\partial r} - \\
&- \frac{i}{(-g^{00})}\tilde{H}_0^0\tilde{H}_2^2\gamma_0\gamma_2\frac{\partial}{\partial\theta} - \frac{i}{(-g^{00})}\tilde{H}_0^0\tilde{H}_3^3\gamma_0\gamma_3\frac{\partial}{\partial\varphi} - \frac{i}{(-g^{00})}\tilde{H}_0^0\tilde{H}_0^3\frac{\partial}{\partial\varphi}.
\end{aligned} \tag{98}$$

For concerned metric  $\tilde{H}$  in Eq. (2) is not equal to zero

$$\Delta\tilde{H} = \frac{i}{4}\left(\frac{\partial\tilde{H}_{30}}{\partial r} + \frac{g^{03}}{g^{00}}\frac{\partial\tilde{H}_{33}}{\partial r}\right)\tilde{H}_1^1S_{13} + \frac{i}{4}\left(\frac{\partial\tilde{H}_{30}}{\partial\theta} + \frac{g^{03}}{g^{00}}\frac{\partial\tilde{H}_{33}}{\partial\theta}\right)\tilde{H}_2^2S_{23} \tag{99}$$

The Hamiltonian  $H_\eta$  is calculated using formula (2).

$$\begin{aligned}
H_\eta &= \frac{im}{(-g^{00})}\tilde{H}_0^0\gamma_0 - \frac{i}{(-g^{00})}\tilde{H}_0^0\tilde{H}_1^1\gamma_0\gamma_1\left(\frac{\partial}{\partial r} + \frac{1}{r}\right) - \\
&- \frac{i}{(-g^{00})}\tilde{H}_0^0\tilde{H}_2^2\gamma_0\gamma_2\left(\frac{\partial}{\partial\theta} + \frac{1}{2}\text{ctg}\theta\right) - \frac{i}{(-g^{00})}\tilde{H}_0^0\tilde{H}_3^3\gamma_0\gamma_3\frac{\partial}{\partial\varphi} - \\
&- \frac{i}{2}\gamma_0\gamma_1\left[\frac{\partial}{\partial r}\frac{\tilde{H}_0^0\tilde{H}_1^1}{(-g^{00})}\right] - \frac{i}{2}\gamma_0\gamma_2\left[\frac{\partial}{\partial\theta}\frac{\tilde{H}_0^0\tilde{H}_2^2}{(-g^{00})}\right] - \frac{i}{(-g^{00})}\tilde{H}_0^0\tilde{H}_0^3\frac{\partial}{\partial\varphi} - \\
&- \frac{i}{4}\gamma_3\gamma_1\tilde{H}_1^1\left(\frac{\partial\tilde{H}_{30}}{\partial r} + \frac{g^{03}}{g^{00}}\frac{\partial\tilde{H}_{33}}{\partial r}\right) + \frac{i}{4}\gamma_2\gamma_3\tilde{H}_2^2\left(\frac{\partial\tilde{H}_{30}}{\partial\theta} + \frac{g^{03}}{g^{00}}\frac{\partial\tilde{H}_{33}}{\partial\theta}\right).
\end{aligned} \tag{100}$$

We put the tetrad vector components

$$\tilde{H}_0^0 = \sqrt{(-g^{00})}, \quad \tilde{H}_1^1 = \frac{\sqrt{\Delta}}{\rho}, \quad \tilde{H}_2^2 = \frac{1}{\rho},$$

$$\begin{aligned}\tilde{H}_{\underline{3}}^3 &= \frac{1}{\sin \theta \cdot \sqrt{\Delta} \sqrt{(-g^{00})}}, \quad \tilde{H}_{\underline{0}}^3 = \frac{ar_0 r}{\rho^2 \sqrt{-g^{00}}}, \\ \tilde{H}_{\underline{3}}^0 &= -\frac{ar_0 r \sin \theta}{\rho^2 \sqrt{\Delta} \sqrt{(-g^{00})}}, \quad \tilde{H}_{\underline{2}}^3 = \sin \theta \cdot \sqrt{\Delta} \sqrt{(-g^{00})}\end{aligned}$$

and the metric component

$$g^{00} = -\frac{1}{\Delta} \left( r^2 + a^2 + \frac{a^2 r_0 r}{\rho^2} \sin^2 \theta \right), \quad g^{03} = -\frac{ar_0 r}{\Delta \rho^2} \quad (101)$$

into (100). Finally,

$$\begin{aligned}H_\eta &= \frac{im}{\sqrt{(-g^{00})}} \gamma_{\underline{0}} - \frac{i\sqrt{\Delta}}{\rho \sqrt{(-g^{00})}} \gamma_{\underline{0}} \gamma_{\underline{1}} \left( \frac{\partial}{\partial r} + \frac{1}{r} \right) - \\ &- \frac{i}{\rho \sqrt{(-g^{00})}} \gamma_{\underline{0}} \gamma_{\underline{2}} \left( \frac{\partial}{\partial \theta} \frac{1}{2} \operatorname{ctg} \theta \right) - \frac{i}{\sin \theta \cdot (-g^{00}) \sqrt{\Delta}} \gamma_{\underline{0}} \gamma_{\underline{3}} \frac{\partial}{\partial \varphi} - \\ &- \frac{i}{(-g^{00})} \frac{ar_0 r}{\rho^2 \Delta} \frac{\partial}{\partial \varphi} - \frac{i}{2} \gamma_{\underline{0}} \gamma_{\underline{1}} \left[ \frac{\partial}{\partial r} \frac{\sqrt{\Delta}}{\rho \sqrt{(-g^{00})}} \right] - \\ &- \frac{i}{2} \gamma_{\underline{0}} \gamma_{\underline{2}} \left[ \frac{\partial}{\partial \theta} \frac{1}{\rho \sqrt{(-g^{00})}} \right] - \\ &- \frac{i}{4} \gamma_{\underline{3}} \gamma_{\underline{1}} \frac{\sqrt{\Delta}}{\rho} \left( -\frac{\partial}{\partial r} \frac{ar_0 r \sin \theta}{\rho^2 \sqrt{\Delta} (-g^{00})} + \frac{ar_0 r}{\Delta \rho^2 g^{00}} \frac{\partial}{\partial r} \sin \theta \sqrt{\Delta} \sqrt{(-g^{00})} \right) + \\ &+ \frac{i}{4} \gamma_{\underline{2}} \gamma_{\underline{3}} \frac{1}{\rho} \left( -\frac{\partial}{\partial \theta} \frac{ar_0 r \sin \theta}{\rho^2 \sqrt{\Delta} \sqrt{(-g^{00})}} + \frac{ar_0 r}{\Delta \rho^2 g^{00}} \frac{\partial}{\partial \theta} \sin \theta \sqrt{\Delta} \sqrt{(-g^{00})} \right). \end{aligned} \quad (102)$$

The quantities  $-g^{00}$ ,  $\Delta$ ,  $\rho$  are determined by the expressions (101), (97).

In order to turn to the Schwarzschild Hamiltonian, one should assume that

$$a = 0, \quad \Delta \rightarrow r^2 - r_0 r, \quad \rho \rightarrow r, \quad \left( r^2 + a^2 + \frac{a^2 r_0 r}{\rho^2} \sin^2 \theta \right) \rightarrow r^2. \quad (103)$$

Such a replacement allows us to derive from (102) the Hamiltonian  $H_\eta$  for the Schwarzschild field

$$H_\eta = im\sqrt{f}\gamma_0 - i\sqrt{f}\gamma_0 \left\{ \gamma_1\sqrt{f} \left( \frac{\partial}{\partial r} + \frac{1}{r} \right) + \gamma_2 \frac{1}{r} \left( \frac{\partial}{\partial \theta} + \frac{1}{2} \text{ctg } \theta \right) + \right. \\ \left. + \gamma_3 \frac{1}{r \cdot \sin \theta} \frac{\partial}{\partial \varphi} \right\} - \frac{i}{2} \frac{\partial f}{\partial r} \cdot \gamma_0 \gamma_1. \quad (104)$$

If we confine ourselves to summands no above the first order of smallness by parameters  $\frac{r_0}{r}, \frac{ar_0}{r^2}$  in expression for  $H_\eta$  (102), we obtain the self-conjugate Hamiltonian for the Kerr weak field.

$$H_\eta^{app} = im \left( 1 - \frac{r_0}{2r} \right) \gamma_0 - i \left( 1 - \frac{r_0}{r} \right) \gamma_0 \gamma_1 \left( \frac{\partial}{\partial r} + \frac{1}{r} \right) - \\ - i \left( 1 - \frac{r_0}{2r} \right) \gamma_0 \left\{ \gamma_2 \frac{1}{r} \left( \frac{\partial}{\partial \theta} + \frac{1}{2} \text{ctg } \theta \right) + \gamma_3 \frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi} \right\} - \frac{ir_0}{2r^2} \gamma_0 \gamma_1 - \\ - i \frac{ar_0}{r^3} \frac{\partial}{\partial \varphi} - i \frac{3}{4} \gamma_3 \gamma_1 \frac{ar_0}{r^3} \sin \theta. \quad (105)$$

In Section 7 the expression (105) will be obtained by algorithm of paper [1] and also with use of Eq. (1). Previously the self-conjugate Hamiltonian for the Kerr weak field was derived in papers [16], [17] at record of a metric in isotropic coordinates.

## 7. Weak axial symmetric gravitational field

### 7.1. The Kerr metric in coordinates of the Boyer-Lindquist

For our tasks we write the expression (94) - (97) with terms no above the first order of smallness by parameters  $\frac{r_0}{r}$  and  $\frac{r_0 a}{r^2}$ . In this approximation

$$\Delta \approx r^2 \left( 1 - \frac{r_0}{r} \right); \rho \approx r; (-g^{00}) \approx \left( 1 + \frac{r_0}{r} \right) \quad (106)$$

$$\begin{aligned}
g^{\alpha\beta} &= \\
&= \begin{array}{|c|c|c|c|} \hline -\left(1 - \frac{r_0}{r}\right) & 0 & 0 & -\frac{ar_0}{r} \sin^2 \theta \\ \hline 0 & \left(1 + \frac{r_0}{r}\right) & 0 & 0 \\ \hline 0 & 0 & r^2 & 0 \\ \hline -\frac{ar_0}{r} \sin^2 \theta & 0 & 0 & r^2 \cdot \sin^2 \theta \\ \hline \end{array} . \tag{107}
\end{aligned}$$

$$\begin{aligned}
g^{\alpha\beta} &= \\
&= \begin{array}{|c|c|c|c|} \hline -\left(1 + \frac{r_0}{r}\right) & 0 & 0 & -\frac{ar_0}{r^3} \\ \hline 0 & \left(1 - \frac{r_0}{r}\right) & 0 & 0 \\ \hline 0 & 0 & \frac{1}{r^2} & 0 \\ \hline -\frac{ar_0}{r^3} & 0 & 0 & \frac{1}{r^2 \cdot \sin^2 \theta} \\ \hline \end{array} . \tag{108}
\end{aligned}$$

$$g = -r^4 \cdot \sin^2 \theta; g_G = -1; \eta = \left(g_G g^{00}\right)^{\frac{1}{4}} = \left(1 + \frac{r_0}{r}\right)^{\frac{1}{4}}. \tag{109}$$

## 7.2. The tetrad vectors in the Schwinger gauge

We will need expressions for tetrad vectors in the Schwinger gauge. The results of calculating the components of tetrad vectors  $\tilde{H}_{\underline{\mu}}^{\alpha}$  are presented in Table 4. Table 5 shows the components of vectors  $\tilde{H}_{\underline{\mu}\alpha}$ .

Table 4: Tetrad vectors  $\tilde{H}_{\underline{\mu}}^{\alpha}$

Tetrad vectors	Tetrad vector components			
$\tilde{H}_{\underline{0}}^{\alpha}$	$\tilde{H}_{\underline{0}}^0 \approx \left(1 + \frac{r_0}{2r}\right)$	$\tilde{H}_{\underline{0}}^1 = 0$	$\tilde{H}_{\underline{0}}^2 = 0$	$\tilde{H}_{\underline{0}}^3 \approx \frac{ar_0}{r^3}$
$\tilde{H}_{\underline{1}}^{\alpha}$	$\tilde{H}_{\underline{1}}^0 = 0$	$\tilde{H}_{\underline{1}}^1 \approx \left(1 - \frac{r_0}{2r}\right)$	$\tilde{H}_{\underline{1}}^2 = 0$	$\tilde{H}_{\underline{1}}^3 = 0$
$\tilde{H}_{\underline{2}}^{\alpha}$	$\tilde{H}_{\underline{2}}^0 = 0$	$\tilde{H}_{\underline{2}}^1 = 0$	$\tilde{H}_{\underline{2}}^2 \approx \frac{1}{r}$	$\tilde{H}_{\underline{2}}^3 = 0$
$\tilde{H}_{\underline{3}}^{\alpha}$	$\tilde{H}_{\underline{3}}^0 = 0$	$\tilde{H}_{\underline{3}}^1 = 0$	$\tilde{H}_{\underline{3}}^2 = 0$	$\tilde{H}_{\underline{3}}^3 \approx \frac{1}{r \sin \theta} \left(1 - \frac{r_0}{2r}\right)$

Table 5: Tetrad vectors  $\tilde{H}_{\underline{\mu}\alpha}$

Tetrad vectors	Tetrad vector components			
$\tilde{H}_{\underline{0}\alpha}$	$\tilde{H}_{\underline{00}} \approx -\left(1 - \frac{r_0}{2r}\right)$	$\tilde{H}_{\underline{01}} = 0$	$\tilde{H}_{\underline{02}} = 0$	$\tilde{H}_{\underline{03}} = 0$
$\tilde{H}_{\underline{1}\alpha}$	$\tilde{H}_{\underline{10}} = 0$	$\tilde{H}_{\underline{11}} \approx \left(1 + \frac{r_0}{2r}\right)$	$\tilde{H}_{\underline{12}} = 0$	$\tilde{H}_{\underline{13}} = 0$
$\tilde{H}_{\underline{2}\alpha}$	$\tilde{H}_{\underline{20}} = 0$	$\tilde{H}_{\underline{21}} = 0$	$\tilde{H}_{\underline{22}} = r$	$\tilde{H}_{\underline{23}} = 0$
$\tilde{H}_{\underline{3}\alpha}$	$\tilde{H}_{\underline{30}} \approx -\frac{ar_0}{r^2} \cdot \sin \theta$	$\tilde{H}_{\underline{31}} = 0$	$\tilde{H}_{\underline{32}} = 0$	$\tilde{H}_{\underline{33}} \approx \frac{r}{\sin \theta} \cdot \left(1 + \frac{r_0}{2r}\right)$

### 7.3. Christoffel symbols

Non-zero Christoffel symbols are:

$$\left[ \begin{array}{l} \left( \begin{array}{c} 0 \\ 01 \end{array} \right) = \frac{1}{2} \frac{r_0}{r^2} \\ \left( \begin{array}{c} 0 \\ 13 \end{array} \right) = -\frac{3}{2} \frac{ar_0}{r^2} \sin^2 \theta \end{array} \right] \left[ \begin{array}{l} \left( \begin{array}{c} 1 \\ 00 \end{array} \right) = \frac{1}{2} \frac{r_0}{r^2} \\ \left( \begin{array}{c} 1 \\ 03 \end{array} \right) = -\frac{ar_0}{2r^2} \sin^2 \theta \\ \left( \begin{array}{c} 1 \\ 11 \end{array} \right) = -\frac{1}{2} \frac{r_0}{r^2} \\ \left( \begin{array}{c} 1 \\ 13 \end{array} \right) = -\frac{3}{2} \frac{ar_0}{r^2} \sin^2 \theta \\ \left( \begin{array}{c} 1 \\ 22 \end{array} \right) = -r \left( 1 - \frac{r_0}{r} \right) \\ \left( \begin{array}{c} 1 \\ 33 \end{array} \right) = -r \sin^2 \theta \left( 1 - \frac{r_0}{r} \right) \end{array} \right] \quad (110)$$

$$\left[ \begin{array}{l} \left( \begin{array}{c} 2 \\ 03 \end{array} \right) = -\frac{ar_0}{r^3} \sin \theta \cos \theta \\ \left( \begin{array}{c} 2 \\ 12 \end{array} \right) = \frac{1}{r} \\ \left( \begin{array}{c} 2 \\ 33 \end{array} \right) = -\sin \theta \cos \theta \end{array} \right] \left[ \begin{array}{l} \left( \begin{array}{c} 3 \\ 01 \end{array} \right) = \frac{ar_0}{2r^4} \\ \left( \begin{array}{c} 3 \\ 02 \end{array} \right) = -\frac{ar_0 \cos \theta}{r^3 \sin \theta} \\ \left( \begin{array}{c} 3 \\ 23 \end{array} \right) = \frac{\cos \theta}{\sin \theta} \end{array} \right].$$

### 7.4. Bispinor connectivities

The bispinor connectivities are calculated using of formula



$$\tilde{\Phi} = \frac{1}{4} \tilde{H}_\mu^\varepsilon \tilde{H}_{\nu\varepsilon;\alpha} S^{\mu\nu}. \quad (111)$$

We obtain:

$$\left. \begin{aligned} \tilde{\Phi}_0 &= \frac{1}{4} \frac{r_0}{r^2} S_{\underline{01}} + \frac{1}{2} \frac{ar_0}{r^3} \cos \theta \cdot S_{\underline{23}} + \frac{1}{4} \frac{ar_0}{r^3} \sin \theta \cdot S_{\underline{31}} \\ \tilde{\Phi}_1 &= -\frac{3}{4} \frac{ar_0}{r^3} \sin \theta \cdot S_{\underline{03}} \\ \tilde{\Phi}_2 &= -\frac{1}{2} \left(1 - \frac{r_0}{2r}\right) S_{\underline{12}} \\ \tilde{\Phi}_3 &= -\frac{3}{4} \frac{ar_0}{r^2} \sin^2 \theta \cdot S_{\underline{01}} - \frac{1}{2} \cos \theta \cdot S_{\underline{23}} + \frac{1}{2} \left(1 - \frac{r_0}{2r}\right) \sin \theta \cdot S_{\underline{31}} \end{aligned} \right\}. \quad (112)$$

### 7.5. The Hamiltonian $H_\eta$

We obtain the expression for  $\tilde{H}$  taking into account 7.1 - 7.4 and using of formula (22)

$$\begin{aligned} \tilde{H} &= im \left(1 - \frac{r_0}{2r}\right) \cdot \gamma_{\underline{0}} - i \left(1 - \frac{r_0}{r}\right) \cdot \gamma_{\underline{0}} \gamma_{\underline{1}} \left(\frac{\partial}{\partial r} + \frac{1}{r}\right) - \\ &- i \left(1 - \frac{r_0}{2r}\right) \frac{1}{r} \cdot \gamma_{\underline{0}} \gamma_{\underline{2}} \left(\frac{\partial}{\partial \theta} + \frac{1}{2} \operatorname{ctg} \theta\right) - i \cdot \frac{ar_0}{r^3} \frac{\partial}{\partial \varphi} - \\ &- i \left(1 - \frac{r_0}{2r}\right) \frac{1}{r \sin \theta} \cdot \gamma_{\underline{0}} \gamma_{\underline{3}} \frac{\partial}{\partial \varphi} - i \cdot \gamma_{\underline{0}} \gamma_{\underline{1}} \frac{r_0}{4r^2} - i \frac{3}{4} \frac{ar_0}{r^3} \sin \theta \cdot S_{\underline{31}} \end{aligned} \quad (113)$$

Since the Kerr solution is stationary, the general formula for  $H_\eta$

$$H_\eta = \tilde{\eta} \tilde{H} \tilde{\eta}^{-1} + i \tilde{\eta} \frac{\partial \tilde{\eta}^{-1}}{\partial t} \quad (114)$$

in our case can be written as

$$H_\eta = \tilde{\eta} \tilde{H} \tilde{\eta}^{-1} \quad (115)$$

where  $\eta$  is determined by relation (109).

Consequently  $H_\eta$  can be written as

$$\begin{aligned}
H_\eta = & im \left(1 - \frac{r_0}{2r}\right) \cdot \gamma_0 - i \left(1 - \frac{r_0}{r}\right) \cdot \gamma_0 \gamma_1 \left(\frac{\partial}{\partial r} - \frac{1}{r}\right) - \\
& -i \left(1 - \frac{r_0}{2r}\right) \frac{1}{r} \cdot \gamma_0 \gamma_2 \left(\frac{\partial}{\partial \theta} + \frac{1}{2} \operatorname{ctg} \theta\right) - \\
& -i \gamma_0 \gamma_1 \frac{r_0}{2r^2} - i \gamma_3 \gamma_1 \frac{3}{4} \frac{ar_0}{r^3} \sin \theta - \\
& -i \left(1 - \frac{r_0}{2r}\right) \frac{1}{r \sin \theta} \cdot \gamma_0 \gamma_3 \frac{\partial}{\partial \theta} - i \frac{ar_0}{r^3} \frac{\partial}{\partial \theta}.
\end{aligned} \tag{116}$$

The expression (116) coincides with the expression (105), which was derived by expansion of the general expression for  $H_\eta$  (102). In turn, the general expression (102) is derived with use of formula (2).

By analogy, we can check validity of formula (1) for concerned metric (107) with use of (113).

Thereby the same expression for the Kerr weak field is derived in fact in three different ways.

By the example of the Kerr metric for block-diagonal metrics, like (3), one can see essential simplification of deriving algorithm of the Dirac self-conjugate Hamiltonians with flat scalar product with use of formula (2).

## 8. The Friedmann open model

Let us concern the case of the Friedmann open model in coordinate

$$(x^0, x^1, x^2, x^3) = (t, \chi, \theta, \varphi)$$

The non-stationary metric for this model is:

$$ds^2 = -dt^2 = a^2(t) \left( d\chi^2 + \operatorname{sh}^2 \chi \left[ d\theta^2 + \sin^2 \theta d\varphi^2 \right] \right) \tag{117}$$

$$g = -a^6 \operatorname{sh}^4 \chi \sin^2 \theta, g_G = -a^6. \tag{118}$$

Non-zero components of tetrad vectors  $\tilde{H}_\alpha^\alpha$  in the Schwinger gauge are:

$$\left. \begin{aligned} &\left\{ \begin{aligned} \tilde{H}_0^0 &= 1, & \tilde{H}_1^1 &= \frac{1}{a}, & \tilde{H}_2^2 &= \frac{1}{a \cdot \text{sh } \chi}, & \tilde{H}_3^3 &= \frac{1}{a \cdot \text{sh } \chi \cdot \sin \theta}, \\ \tilde{H}_0^1 &= -1, & \tilde{H}_1^0 &= a, & \tilde{H}_2^1 &= a \cdot \text{sh } \chi, & \tilde{H}_3^2 &= a \cdot \text{sh } \chi \cdot \sin \theta, \\ \tilde{H}_\alpha^0 &= 1, & \tilde{H}_1^1 &= a, & \tilde{H}_2^2 &= a \cdot \text{sh } \chi, & \tilde{H}_3^3 &= a \cdot \text{sh } \chi \cdot \sin \theta, \\ \tilde{H}^{00} &= -1, & \tilde{H}^{11} &= \frac{1}{a}, & \tilde{H}^{22} &= \frac{1}{a \cdot \text{sh } \chi}, & \tilde{H}^{33} &= \frac{1}{r \cdot \text{sh } \chi \cdot \sin \theta} \end{aligned} \right\}. \end{aligned} \right\} \quad (119)$$

Calculation if the Hamiltonian results in the following:

$$\begin{aligned} \tilde{H} = im\gamma_0 - i\gamma_0 \frac{1}{a} \left\{ \gamma_1 \left( \frac{\partial}{\partial \chi} + \text{cth } \chi \right) + \gamma_2 \frac{1}{\text{sh } \chi} \left( \frac{\partial}{\partial \theta} + \frac{1}{2} \text{ctg } \theta \right) + \right. \\ \left. + \gamma_3 \frac{1}{\text{sh } \chi \cdot \sin \theta} \frac{\partial}{\partial \varphi} \right\} - i \frac{3}{2} \frac{\dot{a}}{a}. \end{aligned} \quad (120)$$

The Hamiltonian is determined by formula (2). For this metric  $\Delta \tilde{H} = 0$

$$\begin{aligned} \tilde{H} = im\gamma_0 - i\gamma_0 \frac{1}{a} \left\{ \gamma_1 \left( \frac{\partial}{\partial \chi} + \text{cth } \chi \right) + \gamma_2 \frac{1}{\text{sh } \chi} \left( \frac{\partial}{\partial \theta} + \frac{1}{2} \text{ctg } \theta \right) + \right. \\ \left. + \gamma_3 \frac{1}{\text{sh } \chi \cdot \sin \theta} \frac{\partial}{\partial \varphi} \right\}. \end{aligned} \quad (121)$$

In a quasi-stationary approximation for the cosmological time  $t$  the energy operator for particle, moving in  $\chi$  - direction, is:

$$E = \sqrt{H_\eta^2} = \sqrt{m^2 + \frac{\mathbf{p}_\chi^2}{a^2(t)}}. \quad (122)$$

Here

$$\mathbf{p}_\eta = -i \left( \frac{\partial}{\partial \chi} + \text{cth } \chi \right)$$

Let us mark

$$a(t) \operatorname{sh} \chi = \frac{a(t)}{a_0} a_0 \operatorname{sh} \chi = b(t) a_0 \operatorname{sh} \chi = b(t) r, \quad (123)$$

where  $b(t_0) = 1$  ; zero indexes corresponds to present time ( $t \leq t_0$ ) .

If at the present time the radius of the Universe spatial curvature tends to infinity  $a \rightarrow \infty$  , then

$$r \approx a_0 \chi \quad (124)$$

In this case the Hamiltonian (121) for the spatially flat Friedmann model equals

$$\begin{aligned} H_\eta = im\gamma_0 - i\gamma_0\gamma_1 \frac{1}{b(t)} \left( \frac{\partial}{\partial r} + \frac{1}{r} \right) - i\gamma_0\gamma_2 \frac{1}{b(t)r} \left( \frac{\partial}{\partial \theta} + \frac{1}{2} \operatorname{ctg} \theta \right) - \\ - i\gamma_0\gamma_3 \frac{1}{b(t)r \sin \theta} \frac{\partial}{\partial \varphi}. \end{aligned} \quad (125)$$

The expression for  $H_\eta$  in Cartesian coordinate system equals

$$H_\eta = im\gamma_0 - \frac{i}{b(t)} \gamma_0 \gamma_k \frac{\partial}{\partial x^k}. \quad (126)$$

## 9. Metric of the Clifford torus

The metric proposed in [10] in the  $(t, \varphi_1, \varphi_2, z)$  coordinates is given by

$$\begin{aligned} ds^2 = -dt^2 + (\rho_1(z))^2 d\varphi_1^2 + (\rho_2(z))^2 d\varphi_2^2 + \\ + \left[ 1 + \rho_1'(z)^2 + \rho_2'(z)^2 \right] dz^2, \\ g = g_G = -\rho_1^2 \rho_2^2 (1 + \rho_1'^2 + \rho_2'^2). \end{aligned} \quad (127)$$

In formula (127), the prime denotes the derivative with respect to the  $z$  coordinate.

Tetrad vectors in the Schwinger gauge:

$$\begin{aligned}
\tilde{H}_0^0 &= 1, & \tilde{H}_0^1 &= 0, & \tilde{H}_0^2 &= 0, & \tilde{H}_0^3 &= 0; \\
\tilde{H}_1^0 &= 0, & \tilde{H}_1^1 &= \frac{1}{\rho_1}, & \tilde{H}_1^2 &= 0, & \tilde{H}_1^3 &= 0; \\
\tilde{H}_2^0 &= 0, & \tilde{H}_2^1 &= 0, & \tilde{H}_2^2 &= \frac{1}{\rho_2}, & \tilde{H}_2^3 &= 0; \\
\tilde{H}_3^0 &= 0, & \tilde{H}_3^1 &= 0, & \tilde{H}_3^2 &= 0, & \tilde{H}_3^3 &= \frac{1}{\rho_3}.
\end{aligned} \tag{128}$$

In expression (128),  $\rho_3(z) = \sqrt{1 + (\rho'_1(z))^2 + (\rho'_2(z))^2}$ .

In accordance with (37), the “reduced” Hamiltonian  $\tilde{H}_{red}$  equals

$$\tilde{H}_{red} = im\gamma_0 - \frac{i}{\rho_1}\gamma_0\gamma_1\frac{\partial}{\partial\varphi_1} - \frac{i}{\rho_2}\gamma_0\gamma_2\frac{\partial}{\partial\varphi_2} - \frac{i}{\rho_3}\gamma_0\gamma_3\frac{\partial}{\partial z}. \tag{129}$$

For concerned metric  $\Delta\tilde{H} = 0$  in (2).

The Hamiltonian  $H_\eta$  in accordance with (2) equals

$$\begin{aligned}
H_\eta &= \frac{1}{2} (\tilde{H}_{red} + \tilde{H}_{red}^+) = \\
&= im\gamma_0 - \frac{i}{\rho_1}\gamma_0\gamma_1\frac{\partial}{\partial\varphi_1} - \frac{i}{\rho_2}\gamma_0\gamma_2\frac{\partial}{\partial\varphi_2} - \frac{i}{\rho_3}\gamma_0\gamma_3\frac{\partial}{\partial z} + \frac{i}{2}\gamma_0\gamma_3\frac{\rho'_3}{\rho_3^2}.
\end{aligned} \tag{130}$$

## 8. Conclusions

In this paper we develop the algorithm proposed in [1] for constructing self-conjugate Hamiltonians  $H_\eta$  in the  $\eta$ -representation with flat scalar product for describing the dynamics of Dirac particles in arbitrary gravitational fields. It is proven that a Hamiltonian in the  $\eta$ -representation for an arbitrary gravitational field, including a time-dependent one, is a Hermitian part of the initial Dirac Hamiltonian  $\tilde{H}$  derived at use of tetrad vectors in the Schwinger gauge. We also prove that for block-diagonal metrics, like (3), the Hamiltonian  $H_\eta$  can be calculated using formula (2) with “reduced” parts of the Hamiltonians  $\tilde{H}$  and  $\tilde{H}^+$  without or with small numbers of summands with bispinor connectivities. Using this procedure, we

for the first time find self-conjugate Hamiltonians  $H_\eta$  for the Kerr metric in coordinates of the Boyer-Lindquist, the Eddington-Finkelstein, Painleve-Gullstrand, Finkelstein-Lemaitre, Kruskal metrics and metrics of the Clifford torus geometry.

The derived expressions for Hamiltonians  $H_\eta$  can be employed to address the issues of stationary energy levels for Dirac particles in the vicinity of black holes, and also at study of scattering and absorption of such particles by black holes.

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